JOURNAL OF APPROXIMATION THEORY 38, 71–80 (1983)

Continuous Selections and Maximal Alternators for Spline Approximation

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Communicated by E. W. Cheney

Received November 13, 1981

1. INTRODUCTION

Recently, Nürnberger and Sommer [3] have shown that there exists a continuous selection for uniform best approximation by splines with fixed knots if and only if the number of knots k is less than or equal to the order m of the splines (for definitions, see the end of this section). We begin this paper by applying a general result on continuous selections [1] to describe the set of all continuous selections for spline approximation, and as a consequence establish that this set is never a singleton.

In Section 3 we examine a certain maximal alternator as a natural candidate for a continuous selection (different from the one constructed in [3]). We show that this maximal alternator is unique, but unfortunately does not provide a continuous selector. On the other hand, we do show the surprising fact that for every function f, the maximal alternator is the value of some continuous selection at f. We conclude the paper with two sections including examples and remarks.

We devote the remainder of this section to notation and basic definitions. Let $[a, b]$ be a closed interval, let k be a positive integer, and suppose

* Supported in part by a grant from Fundacão de Amparo à Pesquisa do Estado de São Paulo.

0021-9045/83 $3.00

$\Delta = \{a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b\}$ is a partition of $[a, b]$. Given an integer $m \geqslant 2$, we define

$$\mathcal{S}_m(\Delta) = \{s \in C^{m-2}[a, b] : s|[x_i, x_{i+1}] \text{ is a polynomial of order } m \text{ for each } i = 0, 1, \ldots, k\}. \tag{1}$$

This is the space of *polynomial splines of order m with (simple) knots* $x_1, \ldots, x_k$. It is well known (cf., e.g., [7]) that $\mathcal{S}_m(\Delta)$ is an $(m+k)$-dimensional Weak–Tschebyscheff space.

For any $f \in C[a, b]$, we denote the *set of uniform best approximations of f by elements of* $\mathcal{S}_m(\Delta)$ by

$$P(f) = \{s \in \mathcal{S}_m(\Delta) : \|f - s\| = d(f)\}, \tag{2}$$

where $d(f) = \operatorname{dist}(f, \mathcal{S}_m(\Delta)) = \inf_{s \in \mathcal{S}_m(\Delta)} \|f - s\|$ and $\| \;\|$ is the usual maximum norm on $C[a, b]$. We are interested in the question of when there exists a continuous mapping

$$S : C[a, b] \to C[a, b]$$

with the property that $Sf \in P(f)$ for all $f \in C[a, b]$. Such an S is called a *continuous selection for the set-valued metric projection P.*

2. The Set of Continuous Selections

Suppose $k \leqslant m$, and let S^* be the continuous selection for spline approximation constructed by Nürnberger and Sommer [3]. Let $f \in C[a, b]$, and given any nonempty subset $A \subseteq P(f)$, define

$$E(f - A) = \{x \in [a, b] : |(f - g)(x)| = d(f) \text{ for all } g \in A\}.$$

Let $P_0 = P(f)$. We now define a sequence $I_0, I_1, \ldots,$ of intervals and a sequence of sets $P_1, \ldots,$ as follows: for each $j = 0, 1, \ldots,$ let

$$I_j = \text{the smallest knot interval which contains } E(f - P_j) \text{ in its relative (with respect to } [a, b]) \text{ interior}$$

and

$$P_{j+1} = \{g \in P_j : g \text{ coincides with } S^*f \text{ on } I_j\},$$

where by a knot interval we mean any interval of the form $[x_l, x_{r+1}]$ with $0 \leqslant l \leqslant r \leqslant k$.

Obviously, there is a smallest j such that all elements of P_j coincide with

S^*f on I_j. Define $I^*(f) = I_j$ and $P^*(f) = P_j$. It follows immediately that $E(f - P^*(f))$ is contained in the relative interior of $I^*(f)$, and it is easy to see that $I^*(f)$ is contained in every knot interval I with the property that $E(f - P_I(f))$ is contained in the relative interior of I, where $P_I(f) = \{g \in P(f): g$ coincides with S^*f on $I\}$. Indeed, if I is such an interval, then

$$\begin{aligned} &P_I(f) \subseteq P_0 \Rightarrow E(f - P_I(f)) \supseteq E(f - P_0) \Rightarrow I_0 \subseteq I \Rightarrow \\ &P_I(f) \subseteq P_1 \Rightarrow E(f - P_I(f)) \supseteq E(f - P_1) \Rightarrow I_1 \subseteq I \Rightarrow \\ &P_I(f) \subseteq P_2 \Rightarrow \cdots I^*(f) \subseteq I. \end{aligned}$$

We can now describe the set of all continuous selections for the metric projection P associated with $\mathscr{S}_m(\Delta)$. By the main result of [1], we have

$$P^*(f) = \{Sf: S \text{ is a continuous selection for the metric projection } P\}. \tag{3}$$

Several examples of this construction are given in Section 4.

As an application of this characterization, we can now show that the continuous selection S^* of [3] is never the only continuous selection for spline approximation with $k \leqslant m$.

THEOREM 1. *There is never a unique continuous selection for spline approximation.*

Proof. By [3], if $k > m$, there is no continuous selection. Suppose now that $k \leqslant m$. Let $f \in C[a, b]$ be a function of norm 1 which alternates m times on the interval $[a, x_1]$ and which vanishes identically on $[x_1, b]$. Clearly, $d(f) = 1$, and all best approximations of f from $\mathscr{S}_m(\Delta)$ are identically zero on $[a, x_1]$. Thus, any spline of the form $\alpha(x - x_1)_+^{m-1}$ with $|\alpha(b - x_1)^{m-1}| \leqslant 1$ is also a best approximation of f. Since $E(f - P(f))$ is contained in $[a, x_1)$, it follows immediately that $I^*(f) = [a, x_1]$ and $P^*(f) = P(f)$. For an example of an f of this type see Example 9. ∎

3. MAXIMAL ALTERNATORS

We need the classical concept of alternation. A nonzero function $g \in C[a, b]$ is said to *alternate p times* (*on p* + 1 *points*) provided there exist $a \leqslant t_1 < \cdots < t_{p+1} \leqslant b$ such that either

$$(-1)^i g(t_i) = \|g\|, \qquad i = 1, 2, \ldots, p + 1,$$

or

$$(-1)^{i+1} g(t_i) = \|g\|, \qquad i = 1, 2, \ldots, p + 1.$$

The following facts about approximation of $f \in C[a, b] \sim \mathcal{S}_m(\Delta)$ are well known (cf. e.g. [5, 6]):

If $s \in \mathcal{S}_m(\Delta)$ is a spline such that $f - s$ alternates $m + k$ times, then s is *a* best approximation of f from $\mathcal{S}_m(\Delta)$. (4)

There exists an interval $[x_i, x_{j+1}]$ with $0 \leqslant i \leqslant j \leqslant k$ in which all best spline approximations s of f agree, and in which $f - s$ alternates at least $m + j - i$ times. (5)

There exists some $s \in P(f)$ such that $f - s$ alternates at least $m + k$ times and $f - s$ does not alternate more times for any other $\tilde{s} \in \mathcal{S}_m(\Delta)$. (6)

A spline $s \in \mathcal{S}_m(\Delta)$ for which $f - s$ alternates a maximal number of times is called a *maximal alternator* for f. Assertion (6) says that for every $f \notin \mathcal{S}_m(\Delta)$, there always exists at least one maximal alternator.

We now show that every $f \in C[a, b] \sim \mathcal{S}_m(\Delta)$ has a unique maximal alternator if and only if $k \leqslant m$. First we need

LEMMA 2. *Suppose that $f \in C[a, b] \sim \mathcal{S}_m(\Delta)$, and that s is a maximal alternator for f. Then for all integers $0 \leqslant l \leqslant r \leqslant k$, the sets*

$$A_r = \{t : a \leqslant t \leqslant x_{r+1} \text{ and } |(f - s)(t)| = d(f)\}$$

$$B_l = \{t : x_l \leqslant t \leqslant b \text{ and } |(f - s)(t)| = d(f)\}$$

are both nonempty. Moreover, if $l > 0$, then there exist points $a \leqslant t_1 < \cdots < t_l < x_l$ with

$$\begin{gathered} (-1)^i (f - s)(t_i) = (-1)^{l-1} (f - s)(\underline{t}), \qquad i = 1, \dots, l, \\ \underline{t} = \inf\{t \in B_l\}. \end{gathered} \tag{7}$$

Similarly, if $r < k$, then there exist points $x_{r+1} < t_{r+1} < \cdots < t_k \leqslant b$ with

$$\begin{gathered} (-1)^i (f - s)(t_i) = (-1)^r (f - s)(\bar{t}), \qquad i = r + 1, \dots, k, \\ \bar{t} = \sup\{t \in A_r\}. \end{gathered} \tag{8}$$

Proof. We claim that $|(f - s)(t)| = d(f)$ for some $x_k < t \leqslant b$. Indeed, if this is not the case, then by choosing an appropriate c, we can construct a spline $\tilde{s} = s + c(x - x_k)_+^{m-1}$ such that $f - \tilde{s}$ alternates at least one more time than $f - s$, contradicting the maximality of s. Thus $B_k \neq \varnothing$ and it follows immediately that $B_l \neq \varnothing$ for all $l = 0, 1, \dots, k$. A similar argument establishes that $A_r \neq \varnothing$ for all $r = 0, 1, \dots, k$.

We turn now to the second part of the lemma. We consider the case

$r < k$—the case $l > 0$ is similar. Suppose $r < k$, but that there are no points $x_{r+1} < t_{r+1} < \cdots < t_k \leqslant b$ satisfying (8). We now show that in this case we can again construct a spline $\tilde{s}$ such that $f - \tilde{s}$ alternates more times than $f - s$, contradicting the hypothesis that s is a maximal alternator. We distinguish two cases.

Case 1. Suppose $\bar{t} = x_{r+1}$. Let $g = (f-s)|I$, where $I = [x_{r+1}, b]$. Since $U = \operatorname{span}\{(x - x_i)_+^{m-1}\}_{i=r+1}^k$ is a Weak–Tschebyscheff space on I, by [2] there is a $u(x) = \sum_{i=r+1}^k c_i(x - x_i)_+^{m-1}$ which is a best approximation of g from U and which is such that $g - u$ alternates at least $k - r$ times on I. (Since $|g(x_{r+1})| = |(f-s)(\bar{t})| = d(f)$, the distance of g from U is $d(f)$). This implies that there exist points $x_{r+1} = t_r < \cdots < t_k \leqslant b$ with

$$(-1)^i (g-u)(t_i) = (-1)^r (f-s)(\bar{t}), \qquad i = r, \ldots, k. \tag{9}$$

But then $\tilde{s} = s + u$ is a best approximating spline for f for which $f - \tilde{s}$ alternates more times than $f - s$. This contradiction shows that the asserted t's must exist.

Case 2. Suppose $\bar{t} < x_{r+1}$. Then $\delta = d(f) - |(f-s)(x_{r+1})| > 0$. Let $c = \sup\{c_{r+1} : \|\sum_{i=r+1}^k c_i(x - x_i)_+^{m-1}\|_I \leqslant 2d(f)\}$, and choose $0 < \varepsilon < x_{r+2} - x_{r+1}$ such that $c\varepsilon^{m-1} < \delta/2$ and $|(f-s)(x)| < d - \delta/2$ for all $x \in I_\varepsilon$, where $I_\varepsilon = [x_{r+1}, x_{r+1} + \varepsilon]$. Now define g on I by

$$\begin{aligned} g(x) &= (f-s)(\bar{t}), && x = x_{r+1}, \\ &= \text{linear}, && x_{r+1} \leqslant x \leqslant x_{r+1} + \varepsilon, \\ &= (f-s)(x), && x_{r+1} + \varepsilon \leqslant x \leqslant b. \end{aligned}$$

By [2], there exists a best approximation u of g from U which alternates at least $k - r$ times on I. In particular, since $|g(x_{r+1})| = |(f-s)(\bar{t})| = d(f)$, there exist points $x_{r+1} = t_r < \cdots < t_k \leqslant b$ where (9) is satisfied. Now by the choice of ε, we have $\|u\|_{I_\varepsilon} \leqslant \delta/2$, while for all $x \in I_\varepsilon$,

$$g(x) > -d + \delta/2 \qquad \text{if} \quad (f-s)(\bar{t}) > 0$$

or

$$g(x) < d - \delta/2 \qquad \text{if} \quad (f-s)(\bar{t}) < 0.$$

We conclude that $x_{r+1} + \varepsilon \leqslant t_{r+1}$. But then since $g(x) = (f-s)(x)$ for $x_{r+1} + \varepsilon \leqslant x \leqslant b$, it follows that $\tilde{s} = s + u$ is a spline for which $f - \tilde{s}$ alternates more times than $f - s$, contradicting the hypothesis that s is a maximal alternator. We have shown that the asserted t's must exist. ∎

THEOREM 3. *The space $\mathscr{S}_m(\Delta)$ has the property that every $f \in C[a,b] \sim \mathscr{S}_m(\Delta)$ has a unique maximal alternator if and only if $k \leqslant m$.*

Proof. We consider first the case $k \leqslant m$. Fix $f \in C[a, b]$, and suppose s and $\tilde{s}$ are both maximal alternators for f. Then by (5), there exists some interval where $s \equiv \tilde{s}$. Let $I = [x_l, x_{r+1}]$ be the largest such interval. If I is all of $[a, b]$, we are done. If not, then by Lemma 2, there exist points $a \leqslant t_1 < \cdots < t_l < x_l$ and $x_{r+1} < t_{r+1} < \cdots < t_k \leqslant b$ satisfying (7) and (8). A similar set of points exists for $\tilde{s}$. Now consider $s - \tilde{s}$. Counting zeros as in [7], we easily see that $s - \tilde{s}$ has l zeros on $[a, x_l)$, an $m + r - l$-tuple interval zero on I, and $k - r$ zeros on $(x_{r+1}, b]$. Since $k \leqslant m$, $s - \tilde{s}$ cannot vanish on any interval other than I, and hence the zeros on $[a, x_l)$ and $(x_{r+1}, b]$ are isolated. Thus $s - \tilde{s}$ has a total of $m + k$ zeros, which by [7, Theorem 4.53] implies that $s \equiv \tilde{s}$. This shows that the maximal alternator is unique.

We turn now to the converse. Suppose that $k > m$. Let $a = y_1 = \cdots = y_m$ and $y_{m+k+1} = \cdots = y_{k+2m} = b$. Set $y_{m+i} = x_i$, $i = 1,..., k$. Let $\{B_i\}_1^{m+k}$ be the corresponding B-spline basis for $\mathscr{S}_m(\Delta)$, (cf. [7]). We now consider B_{m+1}. It is positive on the interval (y_{m+1}, y_{2m+1}) and vanishes identically on the (since $k > m$) nontrivial intervals $L = [a, y_{m+1}]$ and $R = [y_{2m+1}, b]$. Define

$$B = B_{m+1}/\|B_{m+1}\|,$$

and let $y_{m+1} < z < y_{2m+1}$ be a point where B attains its maximum.

It is clear that we can construct $f \in C[a, b]$ such that $\|f\| = 1$, f alternates between ± 1 exactly m times on each subinterval $[x_i, x_{i+1}]$ of $L \cup R$, and

$$\begin{aligned} f(x) &= B(x) - 1, \qquad && y_{m+1} \leqslant x \leqslant z, \\ &= 1 - B(x), && z \leqslant x \leqslant y_{2m+1}. \end{aligned} \tag{10}$$

Then $d(f) = 1$, and since each spline reduces to a polynomial on intervals of the form $[x_i, x_{i+1}]$, it is clear from the construction that any best approximation of f from $\mathscr{S}_m(\Delta)$ must vanish identically on $L \cup R$. But a spline which vanishes on these intervals must be a constant multiple of the B-spline B_{m+1}, and in fact we have

$$P(f) = \{\alpha B \colon -1 \leqslant \alpha \leqslant 1\}.$$

Since all of these splines are such that $f - \alpha B$ alternates the same number of times, they are all maximal alternators. We have established the nonuniqueness in the case $k > m$. ∎

Now that we know when the maximal alternator is unique, there is a natural way to define a selection using it. Suppose $k \leqslant m$. Then for each $f \in C[a, b]$ we define

$$\begin{aligned} Mf &= f, \quad \text{if} \quad f \in \mathscr{S}_m(\Delta), \\ &= s_f \qquad \text{otherwise}, \end{aligned} \tag{11}$$

where s_f is the unique maximal alternator associated with f. This selection unfortunately is not continuous as the following example pointed out to us by Nürnberger and Sommer shows:

EXAMPLE 4. We consider approximation on $[0, 2]$ by splines in the space $\mathscr{S}_2(\{1\})$. Let f and f_n be the functions shown in Fig. 1. It is clear that f_n converges to f while $Mf_n = s$ for all n, and hence does not converge to Mf. ∎

Although M is not continuous, it is true that for every function f, Mf is contained in the set $P^*(f)$ defined in (3). To prove this, we first need a lemma. In the remainder of this section, we suppose $k \leqslant m$. In this case for every $f \in C[a, b]$ there exists a unique largest knot interval $I(f)$ on which all elements of $P(f)$ agree.

LEMMA 5. *Suppose $f \notin \mathscr{S}_m(\Delta)$. Then $I(f)$ is the smallest knot interval containing the set $E(f - P(f))$.*

Proof. Let $I(f) = [x_l, x_{r+1}]$, $0 \leqslant l \leqslant r \leqslant k$. Say $r < k$. We need to show that $J = E(f - P(f)) \cap (x_{r+1}, b] = \varnothing$. First we show that J contains at most $k - r - 1$ points, and in particular, if $J = \{t_j\}_1^p$, then

$$x_{r+j+1} < t_j, \qquad j = 1, \ldots, p. \tag{12}$$

To show this, let $s \neq \tilde{s}$ be two elements in $P(f)$, and fix $1 \leqslant j \leqslant k - r$. Now consider $e = s - \tilde{s}$ restricted to the interval $[x_r, x_{r+j+1}]$. Clearly $e(t) = 0$ for every $t \in J$. Counting zeros as in [7], e has an m-tuple zero on the interval $[x_r, x_{r+1}]$, and by the definition of $I(f)$ and the fact that $k \leqslant m$, it cannot vanish on any other subinterval of $[x_r, x_{r+j+1}]$. But then since $e \in \mathscr{S}_m(\{x_{r+1}, \ldots, x_{r+j}\})$, by [7, Theorem 4.53], it can have at most $j - 1$ zeros in $(x_{r+1}, x_{r+j+1}]$.

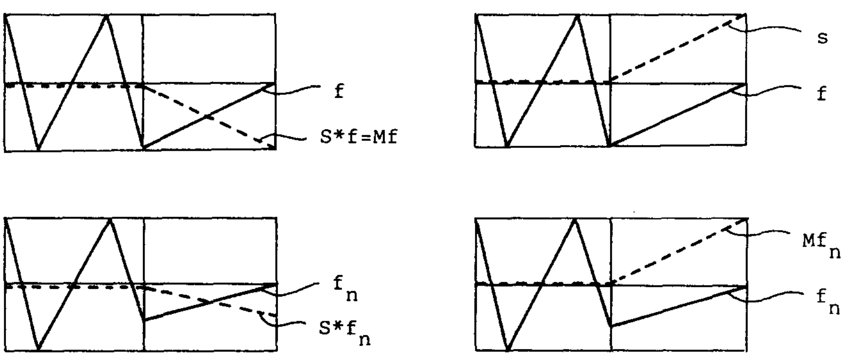

FIGURE 1

We can now show that J is actually empty. Suppose $J = \{t_i\}_1^p$ with $1 \leqslant p \leqslant k - r - 1$. Let $s \in$ rel interior $P(f)$. Then $|(f-s)(t)| = d(f)$ for $t \in (x_r, b]$ only at $t \in J$. By (12) and [7, Theorem 4.78], there exists $g \in \operatorname{span}\{(x - x_i)_+^{m-1}\}_{i=r+2}^{r+p+1}$ such that $g(t_i) = \operatorname{sgn}(f-s)(t_i)$, $i = 1,..., p$. But then for sufficiently small $\varepsilon > 0$, the spline $\tilde{s} = s - \varepsilon g \in P(f)$, and $f - \tilde{s}$ has no extreme points in $(x_{r+1}, b]$. This contradiction of the definition of J implies that J must be empty.

We have shown that $I(f)$ is a knot interval containing $E(f - P(f))$. We now show that it is the smallest one. It suffices to show that $E(f - P(f)) \cap (x_r, x_{r+1}] \neq \varnothing$ and $E(f - P(f)) \cap [x_l, x_{l+1}) \neq \varnothing$. Suppose $E(f - P(f)) \cap (x_r, x_{r+1}] = \varnothing$. Then if $s \in$ rel interior $P(f)$, and if $\varepsilon \neq 0$ is sufficiently small, $s + \varepsilon(x - x_r)_+^{m-1} \in P(f)$, contradicting the fact that all elements of $P(f)$ have to agree with s on $(x_r, x_{r+1}] \subseteq I(f)$. The other case is similar. ∎

THEOREM 6. *For every* $f \in C[a, b]$, $Mf \in P^*(f)$.

Proof. By the definition of $P^*(f)$, it suffices to show that $Mf \in P_i$ for $i = 1, 2,...$. Suppose $Mf \in P_i$. We now prove that $Mf \in P_{i+1}$. As in Lemma 5, there exists a unique largest knot interval $[x_l, x_{r+1}]$, $0 \leqslant l \leqslant r \leqslant k$, on which all elements of P_i agree, and the same proof as in the lemma shows that $E(f - P_i) \subset [x_l, x_{r+1}]$. Since $Mf \in P_i$, we know that $Mf = S^*f$ on $[x_l, x_{r+1}]$. To show that $Mf \in P_{i+1}$, it suffices to show that $Mf = S^*f$ in a neighborhood of $E\ (f - P_i)$. If $r < k$ and $x_{r+1} \in E(f - P_i)$, then since by Lemma 2, $f - Mf$ alternates at least $k - r$ times on $[x_{r+1}, b]$, it follows from the construction of S^* that $S^*f = Mf$ on $[x_{r+1}, x_{r+2}]$. Similarly, if $0 < l$ and $x_l \in E(f - P_i)$, then $S^*f = Mf$ on $[x_{l-1}, x_l]$. This completes the proof. ∎

This theorem implies that in the construction of $P^*(f)$ defined in Section 2, one can replace S^*f by Mf (cf. the examples in Section 4).

4. EXAMPLES

In this section we give three examples of approximation of functions in $C[0, 3]$ by splines in the space $\mathscr{S}_2(\Delta)$ with $\Delta = \{1, 2\}$.

EXAMPLE 7. In the situation in Fig. 2, we have $P_0 = P(f) =$

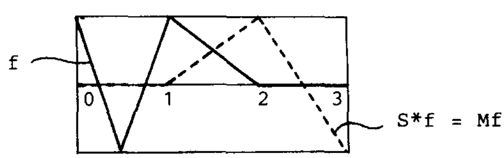

FIGURE 2

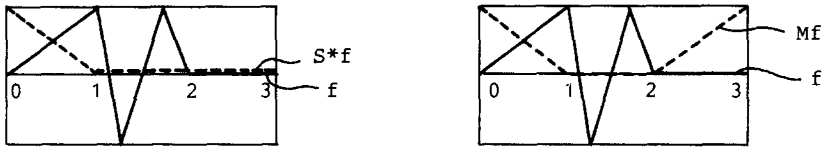

FIGURE 3

$\{\alpha(x-1)_+ + \beta(x-2)_+ : -1 \leqslant \alpha \leqslant 1, -1 \leqslant 2\alpha + \beta \leqslant 1\}$, while $I^*(f) = I_2 = [0, 3]$ and $P^*(f) = P_2 = \{S^*f\}$. ∎

EXAMPLE 8. In the situation in Fig. 3, we have $P_0 = P^*(f) = \{\alpha(1-x) + \alpha(x-1)_+ + \beta(x-2)_+ : -1 \leqslant \alpha, \beta \leqslant 1\}$, while $I^*(f) = I_1 = [0, 2]$ and $P^*(f) = P_1 = \{(1-x) + (x-1)_+ + \beta(x-2)_+ : -1 \leqslant \beta \leqslant 1\}$. ∎

EXAMPLE 9. In the situation in Fig. 4, we have $P(f) = \{\alpha(x-1)_+ + \beta(x-2)_+ : -1 \leqslant \alpha \leqslant 1$ and $-1 \leqslant 2\alpha + \beta \leqslant 1\}$, and $I^*(f) = I_0 = [0, 1]$, and $P^*(f) = P(f)$. ∎

5. REMARKS

(1) The idea of a maximal alternator was introduced in Schumaker [5], where the existence of at least one was established. The fact that maximal alternators are not unique in general was also observed there, and in fact, the idea for the construction of the function f in the proof of Theorem 3 is inherent in [5, Example (3.15)]. Theorem 3 has recently been proved independently by Nürnberger and Sommer [4] using an entirely different proof.

(2) Although we have restricted our attention to splines with simple knots, all of the results of this paper have obvious analogs for the spaces $\mathscr{S}(P_m; M; \Delta)$ of polynomial splines with multiple knots, and for spaces of Tschebyscheffian splines (cf. [5–7]) as well.

(3) The nonuniqueness of continuous selections expressed in Theorem 1

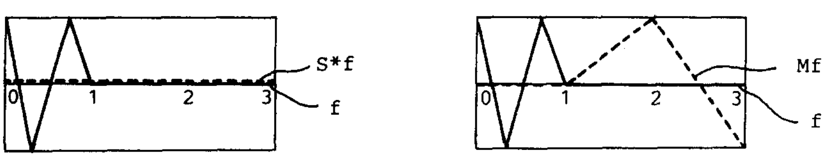

FIGURE 4

coupled with the fact that the continuous selection S^* constructed in [3] is quite complicated suggests the following problem: find a simple continuous selection for spline approximation.

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